Notes 2a

MORE ON IMPROPER INTEGRALS

Suppose $f\colon [a,b) \to \mathbb{R}$ is a function that is Riemann integrable on $[a,c]$ for all $c \in [a,b)$. (Here b could be finite or $+\infty$.) We say that the improper integral

$$\int_a^b f$$

converges, if

$$\lim_{c \to b^-} \int_a^c f \text{ exists in } \mathbb{R}.$$

Otherwise we say the improper integral $\int_a^b f$ is divergent. In the first case we define

$$\int_a^b f := \lim_{c \to b^-} \int_a^c f.$$

Similarly we can consider improper integrals, if $f\colon (a,b] \to \mathbb{R}$ is a function that is Riemann integrable on $[c,b]$ for all $c \in (a,b]$. (Here a could be finite or $-\infty$.)

Theorem 2.1 (Cauchy's criterion)**.** *Suppose $f\colon [a,b) \to \mathbb{R}$ is a function that is Riemann integrable on $[a,c]$ for all $c \in [a,b)$. Then the improper integral $\int_a^b f$ exists, if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $b - \delta < c_1 < c_2 < b$, we have*

$$\left| \int_{c_1}^{c_2} f \right| < \varepsilon.$$

Proof. Just note that $\int_a^b f$ converges, if and only if $\lim_{c \to b^-} \int_a^c f$ exists, and if $F\colon [a,b) \to \mathbb{R}$ is a function, then $\lim_{c \to b^-} F(c)$ exists, if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $b - \delta < c_1 < c_2 < b$, we have

$$|F(c_2) - F(c_1)| < \varepsilon.$$

Letting $F(c) = \int_a^c f$ gives the above result. $\square$

Theorem 2.2 (Absolute convergence test)**.** *Suppose $f\colon [a,b) \to \mathbb{R}$ is a function that is Riemann integrable on $[a,c]$ for all $c \in [a,b)$. If the improper integral $\int_a^b |f|$ exists, then the improper integral $\int_a^b f$ exists.*

Proof. Use Cauchy's criterion: Suppose $\int_a^b |f|$ exists. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $b - \delta < c_1 < c_2 < b$, we have

$$\int_{c_1}^{c_2} |f| < \varepsilon.$$

But since

$$\left|\int_{c_1}^{c_2} f\right| \leq \int_{c_1}^{c_2} |f|,$$

this implies

$$\left|\int_{c_1}^{c_2} f\right| < \varepsilon$$

whenever $b - \delta < c_1 < c_2 < b$. Thus Cauchy's criterion again implies $\int_a^b f$ is convergent. □

Theorem 2.3 (Comparison test)**.** *Suppose $f, g\colon [a,b) \to \mathbb{R}$ are Riemann integrable on $[a,c)$ for all $c \in [a,b)$. Suppose also $0 \leq f(x) \leq g(x)$ for all $x \in [a,b)$. Then the improper integral $\int_a^b f$ converges, if the improper integral $\int_a^b g$ converges.*

As a result, if f, g are as in the statement of the comparison test, and if the improper integral $\int_a^b f$ diverges, then the improper integral $\int_a^b g$ also diverges.

Proof of Comparison test. Again use Cauchy's criteria: If the improper integral $\int_a^b g$ converges, then $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $b - \delta < c_1 < c_2 < b$, we have

$$\int_{c_1}^{c_2} g < \varepsilon.$$

But since

$$0 \leq \int_{c_1}^{c_2} f \leq \int_{c_1}^{c_2} g,$$

this implies

$$\left|\int_{c_1}^{c_2} f\right| < \varepsilon$$

whenever $b - \delta < c_1 < c_2 < b$. Thus Cauchy's criterion again implies $\int_a^b f$ is convergent. □

Note that the above proof would also work, if we merely assume that there exists $c_0 \in [a,b)$ such that $0 \leq f(x) \leq g(x)$ for all $x \in [c_0, b)$. Hence we have the following corollary:

Corollary 2.4 (Comparison test)**.** *Suppose $f, g\colon [a,b) \to \mathbb{R}$ are Riemann integrable on $[a,c)$ for all $c \in [a,b)$. Suppose also that there exists $c_0 \in [a,b)$ such that $0 \leq f(x) \leq g(x)$ for all $x \in [c_0, b)$. Then the improper integral $\int_a^b f$ converges, if the improper integral $\int_a^b g$ converges.*

It takes some practice to be able to use these tests well. One thing that is often involved is the following: If $f(x) \geq 0$ and $g(x) > 0$, then one can make the quotient $\frac{f(x)}{g(x)}$ larger, by increasing the numerator $f(x)$, or by decreasing the denominator $g(x)$. For example, try to verify the following inequalities:

$$\frac{1}{x(x+2015)} \le \frac{1}{x^2} \quad \text{if } x > 0$$

$$\frac{|\cos x|}{1+x^2} \le \frac{1}{x^2}$$

$$\frac{1}{x(x-1)} \le \frac{2}{x^2} \quad \text{if } x > 2$$

$$\frac{1+\sqrt{x}}{\sqrt{x^4+x+1}} \le \frac{2}{x^{3/2}} \quad \text{if } x > 1$$

$$\frac{\ln x}{x^2 - x + 4} \le \frac{c}{x^{3/2}} \quad \text{if } x \text{ is sufficiently large}$$

With more practice, you should be able to come up with these estimates, once you are given the left hand side.